Integrability of implicit differential equations

1995 J. Phys. A: Math. Gen. 28 149

(http://iopscience.iop.org/0305-4470/28/1/018)

J. Phys. A: Math. Gen. 28 (1995) 149–163. Printed in the UK

Integrability of implicit differential equations

Giovanna Mendella†‡, Giuseppe Marmo†‡ and Włodzimierz M Tulczyjew‡§

† Dipartimento di Scienze Fisiche, Mostra d'Oltremare, pad. 19, 80125 Napoli, Italy
‡ Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Italy
§ Dipartimento di Matematica e Fisica, Università di Camerino, 62032 Camerino (MC), Italy

Received 21 March 1994

Abstract. The problem of integrability of differential equations is discussed. Examples and integrability criteria are given. An algorithm for extracting the integrable part of an implicit differential equation is formulated. A procedure for generating a class of submanifolds of the cotangent bundle is defined. This procedure is then used for generating implicit differential equations in the phase space of a mechanical system. Integrability criteria of such equations are established and an extraction algorithm is formulated

1. Introduction

We propose to analyse the problem of integrability of constrained Hamiltonian systems [1] in terms of a representation of these systems as implicit differential equations. In our opinion, this approach offers greater conceptual clarity over the usual representation of constrained Hamiltonian systems as families of Hamiltonian vector fields.

2. Preliminary definitions [2]

The *tangent bundle* of a differential manifold M of dimension m is a differential manifold $\mathsf{T}M$ of dimension $2m$. The underlying set of $\mathsf{T}M$ is the set of equivalence classes of differentiable curves in M called *vectors*. Two curves $\gamma: I \to M$ and $\gamma': I' \to M$ are equivalent if

$$\mathrm{D}^0(f \circ \gamma')(0) = \mathrm{D}^0(f \circ \gamma)(0) \tag{1}$$

and

$$\mathrm{D}^1(f \circ \gamma')(0) = \mathrm{D}^1(f \circ \gamma)(0) \tag{2}$$

for each differentiable function $f: M \to \mathbb{R}$. We use the symbols D^0 and D^1 to denote the zeroth and the first derivative of a function, respectively. The zeroth derivative of a function is the function itself. The sets I and I' are open neighbourhoods of $0 \in \mathbb{R}$. The equivalence class of a curve $\gamma: I \to M$ is denoted by $\mathrm{t}\gamma(0)$. The mapping

$$\begin{aligned} \mathrm{t}\gamma &: \mathbb{R} \to \mathsf{T}M \\ &: s \mapsto \mathrm{t}\gamma(s + \cdot)(0) \end{aligned} \tag{3}$$

is called the *tangent prolongation* of the curve γ. The mapping

$$\begin{aligned}\tau_M &: \mathbf{T}M \to M \\ &: \mathrm{t}\gamma(0) \mapsto \gamma(0)\end{aligned} \tag{4}$$

is called the *tangent bundle projection.*

Let $\eta: M \to N$ be a differentiable mapping. The mapping

$$\mathbf{T}\eta: \mathbf{T}M \to \mathbf{T}N \tag{5}$$

defined by

$$\mathbf{T}\eta(\mathrm{t}\gamma(0)) = \mathrm{t}(\eta \circ \gamma)(0) \tag{6}$$

is called the *tangent mapping* of η. If $\eta: M \to N$ and $\zeta: N \to P$ are differentiable mappings, then

$$\mathbf{T}(\zeta \circ \eta) = \mathbf{T}\zeta \circ \mathbf{T}\eta. \tag{7}$$

A vector $v \in \mathbf{T}M$ is said to be *tangent* to a set $C \subset M$ if there is a representative $\gamma: I \to M$ with $\mathrm{Im}(\gamma) \subset C$. The set of all vectors *tangent* to C is denoted by $\mathbf{T}C$.

We will study only first-order differential equations. Concepts of higher-order differential geometry, introduced below, are useful in the analysis of the integrability of such equations.

The second tangent bundle of a differential manifold M of dimension m is a differential manifold $\mathbf{T}^2M$ of dimension $3m$. The underlying set of $\mathbf{T}^2M$ is the set of equivalence classes of differentiable curves in M. Two curves $\gamma: I \to M$ and $\gamma': I' \to M$ are equivalent if

$$\mathrm{D}^0(f \circ \gamma')(0) = \mathrm{D}^0(f \circ \gamma)(0) \tag{8}$$

$$\mathrm{D}^1(f \circ \gamma')(0) = \mathrm{D}^1(f \circ \gamma)(0) \tag{9}$$

and

$$\mathrm{D}^2(f \circ \gamma')(0) = \mathrm{D}^2(f \circ \gamma)(0) \tag{10}$$

for each differentiable function $f: M \to \mathbb{R}$. The equivalence class of a curve γ is denoted by $\mathrm{t}^2\gamma(0)$. The mapping

$$\begin{aligned}\mathrm{t}^2\gamma &: \mathbb{R} \to \mathbf{T}^2M \\ &: s \mapsto \mathrm{t}^2\gamma(s + \cdot)(0)\end{aligned} \tag{11}$$

is called the *second tangent prolongation* of the curve γ. Two projections are defined by

$$\begin{aligned}\tau_M^{1,2} &: \mathbf{T}^2M \to \mathbf{T}M \\ &: \mathrm{t}^2\gamma(0) \to \mathrm{t}\gamma(0)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\tau_M^2 &: \mathbf{T}^2M \to M \\ &: \mathrm{t}^2\gamma(0) \to \gamma(0).\end{aligned} \tag{13}$$

These projections are related by $\tau_M \circ \tau_M^{1,2} = \tau_M^2$. The projection τ_M^2 is called the *second tangent bundle projection.*

The second tangent bundle will be identified with the set

$$\mathbf{T}^2M = \{w \in \mathbf{TT}M; \mathbf{T}\tau_M(w) = \tau_{\mathbf{T}M}(w)\}. \tag{14}$$

The third tangent bundle $\mathbf{T}^3M$ is defined in a similar way. It is identified with the set

$$\mathbf{T}^3M = \{w \in \mathbf{TTT}M; \mathbf{TT}\tau_M(w) = \mathbf{T}\tau_{\mathbf{T}M}(w) = \tau_{\mathbf{TT}M}(w)\}. \tag{15}$$

It is now clear how higher-order tangent bundles can be defined.

3. Implicit differential equations

Explicit differential equations have been treated extensively in mathematical literature. In contrast, the concept of an implicit differential equation has received little attention. It is barely mentioned in some texts of analysis and is absent from the literature on differential geometry. We are therefore constrained to begin our work with the statement and demonstration of elementary facts about implicit differential equations. Implicit differential equations of interest are those describing the dynamics of a mechanical system in its phase space P. Only the differential manifold structure of the phase space is used in the present section.

An *implicit first-order differential equation* is a submanifold D of the tangent bundle $\mathsf{T}P$ [3].

A curve $\gamma: I \to P$ is called a *solution* of a differential equation $D \subset \mathsf{T}P$ if $\mathrm{t}\gamma(s) \in D$ for each $s \in I$.

An implicit differential equation is said to be *explicit* if it is the image of a differentiable vector field $X: U \to \mathsf{T}P$ defined on an open submanifold $U \subset P$.

An implicit differential equation D is said to be *integrable at* $v \in D$ if there is a solution $\gamma: I \to P$ such that $\mathrm{t}\gamma(0) = v$. An implicit differential equation D is said to be *integrable in a subset* $S \subset D$ if it is integrable at each $v \in S$. An implicit differential equation D is said to be *integrable* if it is integrable at each $v \in D$.

Symmetries and constants of motion are defined in an earlier paper [4].

Integrability of explicit differential equations is well established. Implicit differential equations need not be integrable.

Example 1. Let $P = \mathbb{R}^2$. The tangent bundle $\mathsf{T}P$ is identified with $\mathbb{R}^4$. The implicit differential equation

$$D = \{(x, y, \dot{x}, \dot{y}) \in \mathsf{T}P;\, y = 0, \dot{x} = 0, \dot{y} = x\} \tag{16}$$

is not integrable, except at $(0, 0, 0, 0)$. The only solution of this equation is the constant curve

$$\begin{aligned} \gamma: \mathbb{R} &\to P \\ : s &\mapsto (0, 0). \end{aligned} \tag{17}$$

Example 2. Let $P = \mathbb{R}^2$. The implicit differential equation

$$D = \{(x, y, \dot{x}, \dot{y}) \in \mathsf{T}P;\, y = 0, \dot{x} = 0, \dot{y} > 0\} \tag{18}$$

has no solutions.

Example 3. Let $P = \mathbb{R}^2$. The implicit differential equation

$$D = \{(x, y, \dot{x}, \dot{y}) \in \mathsf{T}P;\, x^2 + y^2 + (\dot{x} - 1)^2 + \dot{y}^2 = 1\} \tag{19}$$

is not integrable at points in the set

$$\{(x, y, \dot{x}, \dot{y}) \in \mathsf{T}P;\, x^2 + y^2 = 1, x \neq 0, \dot{x} = 1, \dot{y} = 0\}. \tag{20}$$

4. Integrability criteria for implicit differential equations

Proposition 1. If $D \subset \mathbf{T}P$ is integrable, then

$$D \subset \mathbf{T}(\tau_P(D)). \tag{21}$$

Proof. We prove below a stronger criterion of integrability. □

Proposition 2. If $D \subset \mathbf{T}P$ is integrable, then

$$D \cap U \subset \mathbf{T}(\tau_P(D \cap U)) \tag{22}$$

for each open submanifold $U \subset \mathbf{T}P$.

Proof. Let $v \in D \cap U$ and let $\gamma : I \to P$ be a solution of D, such that $\mathrm{t}\gamma(0) = v$. The set

$$I' = (\mathrm{t}\gamma)^{-1}(\mathrm{Im}(\mathrm{t}\gamma) \cap U) \tag{23}$$

is an open neighbourhood of $0 \in \mathbb{R}$ and

$$\mathrm{Im}(\mathrm{t}\gamma | I') = \mathrm{Im}(\mathrm{t}\gamma) \cap U \subset D \cap U. \tag{24}$$

It follows that

$$\mathrm{Im}(\gamma | I') \subset \tau_P(D \cap U) \tag{25}$$

and

$$\mathrm{Im}(\mathrm{t}\gamma | I') \subset \mathbf{T}(\tau_P(D \cap U)). \tag{26}$$

Hence, $v = \mathrm{t}\gamma(0) \in \mathbf{T}(\tau_P(D \cap U))$. □

Proposition 3. If $D \subset \mathbf{T}P$ is integrable, then

$$D \cap U \subset \tau_{\mathbf{T}P}(\mathbf{T}(D \cap U) \cap \mathbf{T}^2 P) \tag{27}$$

for each open submanifold $U \subset \mathbf{T}P$.

Proof. Let $v \in D \cap U$ and let $\gamma : I \to P$ be a solution of D, such that $\mathrm{t}\gamma(0) = v$. The set

$$I' = (\mathrm{t}\gamma)^{-1}(\mathrm{Im}(\mathrm{t}\gamma) \cap U) = (\mathrm{t}\gamma)^{-1}(U) \tag{28}$$

is an open neighbourhood of $0 \in \mathbb{R}$ and

$$\mathrm{Im}(\mathrm{t}\gamma | I') = \mathrm{Im}(\mathrm{t}\gamma) \cap U \subset D \cap U. \tag{29}$$

It follows that

$$\mathrm{Im}(\mathrm{t}^2\gamma | I') \subset \mathbf{T}(D \cap U) \cap \mathbf{T}^2 P. \tag{30}$$

Hence, $v = \mathrm{t}\gamma(0) = \tau_{\mathbf{T}P}(\mathrm{t}^2\gamma(0)) \in \tau_{\mathbf{T}P}(\mathbf{T}(D \cap U) \cap \mathbf{T}^2 P)$. □

Proposition 4. If $D \subset \mathsf{T}P$ is integrable, then

$$D \cap U \subset \tau_{\mathsf{T}P}(\tau_{\mathsf{TT}P}(\mathsf{TT}(D \cap U) \cap \mathsf{T}^3 P)) \tag{31}$$

for each open submanifold $U \subset \mathsf{T}P$.

Proof. The proof is similar to that of proposition 3. □

It is obvious that propositions 3 and 4 represent the beginning of an infinite sequence of necessary integrability conditions.

Proposition 5. If $C = \tau_P(D)$ is a submanifold of P and if the mapping

$$\begin{aligned} \tau &: D \to C \\ &: v \mapsto \tau_P(v) \end{aligned} \tag{32}$$

is a surjective submersion, then the condition $D \subset \mathsf{T}C$ is sufficient for integrability of the implicit differential equation $D \subset \mathsf{T}P$.

Proof. Let v be an element of D and let $x = \tau_P(v)$. Let $\sigma: C \to D$ be a (local) section of $\tau: D \to C$ such that $\sigma(x) = v$. If $\varepsilon: D \to \mathsf{T}C$ is the canonical injection, then $X = \varepsilon \circ \sigma: C \to \mathsf{T}C$ is a section of $\tau_C: \mathsf{T}C \to C$ and, hence, a vector field on C. Let $\gamma: I \to C$ be an integral curve of X such that $\gamma(0) = x$. Then $\mathrm{Im}(\mathrm{t}\gamma) \subset D$ and $\mathrm{t}\gamma(0) = X(x) = v$. □

5. Examples

Examples of implicit differential equations in this section have been selected to demonstrate the insufficiency of the integrability criteria, with the exception of proposition 4.

Example 4. Let $P = \mathbb{R}$ and let $\mathsf{T}P$ be identified with $\mathbb{R}^2$. The implicit differential equation

$$D = \{(x, \dot{x}) \in \mathsf{T}P;\ (\dot{x} - a)^2 = x\} \tag{33}$$

is not integrable at $(x, \dot{x}) = (0, a)$ if $a \neq 0$. The criterion of proposition 1 fails at this point. If $a = 0$, then

$$\begin{aligned} \gamma &: \mathbb{R} \to P \\ &: s \mapsto 1 + s + \tfrac{1}{4}s^2 \end{aligned} \tag{34}$$

is a solution of D and $D = \mathrm{Im}(\mathrm{t}\gamma)$. It follows that D is integrable.

Example 5. Let $P = \mathbb{R}$. The implicit differential equation

$$D = \left\{(x, \dot{x}) \in \mathsf{T}P;\ (\dot{x} - a)^3 = x\right\} \tag{35}$$

is the image of the section

$$\begin{aligned} X &: P \to \mathsf{T}P \\ &: x \mapsto (x, x^{1/3} + a). \end{aligned} \tag{36}$$

Since X is not differentiable at $x = 0$, differential equation D is implicit in an essential way. If $a = 0$, then

$$\begin{aligned}\gamma_1 &: \mathbb{R} \to P \\ &: s \mapsto 0\end{aligned} \tag{37}$$

$$\begin{aligned}\gamma_2 &:]0, \infty[\to P \\ &: s \mapsto \left(\tfrac{2}{3}s\right)^{3/2}\end{aligned} \tag{38}$$

and

$$\begin{aligned}\gamma_3 &:]0, \infty[\to P \\ &: s \mapsto -\left(\tfrac{2}{3}s\right)^{3/2}\end{aligned} \tag{39}$$

are solutions of D and $D = \mathrm{Im}(\mathrm{t}\gamma_1) \cup \mathrm{Im}(\mathrm{t}\gamma_2) \cup \mathrm{Im}(\mathrm{t}\gamma_3)$. If $a \neq 0$, then D is not integrable at $(x, \dot{x}) = (0, a)$. (Propositions 1 and 2 are satisfied, proposition 3 is not.)

Example 6. Let $P = \mathbb{R}$ and let $\mathsf{T}P$ be identified with $\mathbb{R}^2$. The implicit differential equation

$$D = \{(x, \dot{x}) \in \mathsf{T}P; \dot{x}^3 - 3\dot{x} = 2x\} \tag{40}$$

is not integrable at $(x, \dot{x}) = (1, -1)$ or $(x, \dot{x}) = (-1, 1)$. (Proposition 1 is satisfied, proposition 2 is not.)

Example 7. Let $P = \mathbb{R}^2$. The tangent bundle $\mathsf{T}P$ is identified with $\mathbb{R}^4$. The implicit differential equation

$$D = \{(x, y, \dot{x}, \dot{y}) \in \mathsf{T}P; (\dot{x} - 1)^3 = x - y + y^2, \dot{x} = \dot{y}\} \tag{41}$$

is not integrable at $(x, y, \dot{x}, \dot{y}) = (0, 0, 1, 1)$. (Propositions 1–3 are satisfied, proposition 4 is not.)

6. Extracting the integrable part of an implicit differential equation

Proposition 5 suggests a method for extracting the integrable part of an implicit differential equation if certain rather strong regularity conditions are satisfied.

Let D be an implicit differential equation in P, let C be the projection $\tau_P(D)$ and let $\tau: D \to C$ be the mapping induced by the restricion of τ_P to D. We construct a sequence of objects

$$(D^0, C^0, \tau^0), (D^1, C^1, \tau^1), \ldots, (D^k, C^k, \tau^k), \ldots \tag{42}$$

where

$$D^0 = D \qquad C^0 = C \qquad \tau^0 = \tau \tag{43}$$

$$D^k = D^{k-1} \cap \mathsf{T}C^{k-1} \qquad C^k = \tau_P(D^k) \tag{44}$$

and $\tau^k: D^k \to C^k$ is the projection induced by the restriction of τ_P to D^k. For each k, it is assumed that the sets C^k are submanifolds and that the mappings τ^k are surjective submersions. Since the dimension of P is finite, the sequence of implicit differential equations

$$D^0, D^1, \ldots, D^k, \ldots \tag{45}$$

is constant, starting with some index $k = m$. The integrable implicit differential equation $D^m \subset \mathsf{T}P$ is the integrable part (possibly empty) of D.

Example 8. Let $\bar{P} = \mathbb{R}^4$. The tangent bundle $\mathsf{T}\bar{P}$ is identified with $\mathbb{R}^8$. Let H be a differentiable function

$$\begin{aligned} H &: \mathbb{R}^2 \to \mathbb{R} \\ &: (x, p) \mapsto H(x, p) \end{aligned} \tag{46}$$

of two real variables. The implicit differential equation

$$\bar{D} = \left\{ (x, p, r, s, \dot{x}, \dot{p}, \dot{r}, \dot{s}) \in \mathsf{T}\bar{P}; r = p, s = 0, \dot{r} = -\frac{\partial H}{\partial x}(x, p), \dot{s} = \dot{x} - \frac{\partial H}{\partial p}(x, p) \right\} \tag{47}$$

is not integrable. We have

$$\bar{D}^0 = \bar{D} \tag{48}$$

$$\bar{C}^0 = \{(x, p, r, s) \in \bar{P}; r = p, s = 0\} \tag{49}$$

$$\mathsf{T}\bar{C}^0 = \{(x, p, r, s, \dot{x}, \dot{p}, \dot{r}, \dot{s}) \in \mathsf{T}\bar{P}; r = p, s = 0, \dot{r} = \dot{p}, \dot{s} = 0\} \tag{50}$$

$$\begin{aligned} \bar{D}^1 = \bar{D}^0 \cap \mathsf{T}\bar{C}^0 = \bigg\{ & (x, p, r, s, \dot{x}, \dot{p}, \dot{r}, \dot{s}) \in \mathsf{T}\bar{P}; r = p, s = 0, \dot{r} = \dot{p} = -\frac{\partial H}{\partial x}(x, p), \\ & \dot{s} = 0, \dot{x} = \frac{\partial H}{\partial p}(x, p) \bigg\} \end{aligned} \tag{51}$$

$$\bar{C}^m = \bar{C}^0 \tag{52}$$

and

$$\bar{D}^m = \bar{D}^1 \tag{53}$$

for $m \geqslant 1$. Regularity conditions are satisfied. The implicit differential equation $\bar{D}^1$ is the integrable part of $\bar{D}$. Let $P = \mathbb{R}^2$ and let σ be the embedding

$$\begin{aligned} \sigma &: P \to \bar{P} \\ &: (x, p) \mapsto (x, p, p, 0). \end{aligned} \tag{54}$$

Then, $\bar{C}^0 = \operatorname{Im}(\sigma)$ and $\bar{D}^1 = \mathsf{T}\sigma(D)$, where

$$D = \left\{ (x, p, \dot{x}, \dot{p}) \in \mathsf{T}P; \dot{x} = \frac{\partial H}{\partial p}(x, p), \dot{p} = -\frac{\partial H}{\partial x}(x, p) \right\} \tag{55}$$

is the image of the vector field

$$\begin{aligned} X &: P \to \mathsf{T}P \\ &: (x, p) \mapsto \left(x, p, \frac{\partial H}{\partial p}(x, p), -\frac{\partial H}{\partial x}(x, p) \right). \end{aligned} \tag{56}$$

Example 9. [5] Let $P = \mathbb{R}^4$ and let the tangent bundle $\mathsf{T}P$ be identified with $\mathbb{R}^8$. The implicit differential equation

$$\begin{aligned} D = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\, & x^2 + y^2 \neq 0,\, p = k[(x^2 + y^2)\dot{x} - (x\dot{x} + y\dot{y})x], \\ & q = k[(x^2 + y^2)\dot{y} - (x\dot{x} + y\dot{y})y],\, \dot{p} = k[(\dot{x}^2 + \dot{y}^2)x - (x\dot{x} + y\dot{y})\dot{x}], \\ & \dot{q} = k[(\dot{x}^2 + \dot{y}^2)y - (x\dot{x} + y\dot{y})\dot{y}]\} \end{aligned} \tag{57}$$

is not integrable. We have

$$D^0 = D \tag{58}$$

$$C^0 = \{(x, y, p, q) \in P;\, x^2 + y^2 \neq 0, xp + yq = 0\} \tag{59}$$

$$\mathsf{T}C^0 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\, x^2 + y^2 \neq 0, xp + yq = 0, \dot{x}p + x\dot{p} + \dot{y}q + y\dot{q} = 0\} \tag{60}$$

$$\begin{aligned} D^1 = D^0 \cap \mathsf{T}C^0 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\, & x^2 + y^2 \neq 0, \\ & p = 0, \dot{p} = 0, q = 0, \dot{q} = 0, \dot{x}y = x\dot{y}\} \end{aligned} \tag{61}$$

$$C^1 = \{(x, y, p, q) \in P;\; x^2 + y^2 \neq 0, p = 0, q = 0\} \tag{62}$$

$$\mathsf{T}C^1 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\, x^2 + y^2 \neq 0, p = 0, \dot{p} = 0, q = 0, \dot{q} = 0\} \tag{63}$$

and

$$D^2 = D^1 \cap \mathsf{T}C^1 = D^1. \tag{64}$$

The implicit differential equation D^1 is the integrable part of D. Solutions of D^1 are curves

$$\begin{aligned} \gamma &: \mathbb{R} \to P \\ &: t \mapsto (af(t), bf(t), 0, 0) \end{aligned} \tag{65}$$

where a and b are numbers and f is a differentiable function such that

$$(af(t))^2 + (bf(t))^2 \neq 0.$$

7. Affine subbundles of T^*P

The symplectic structure of the phase space of a mechanical system makes it possible to generate implicit differential equations from simpler objects. A constrained Hamiltonian system is generated by a function defined on the constraint submanifold. A Dirac system modified by the extraction algorithm is no longer generated in such a simple manner. Hence, we find it necessary to generalize the generation mechanism in order to be able to generate a class of implicit differential equations which includes Dirac systems as well as systems obtained by applying the extraction algorithm. We observe that the generation of a Dirac system from the Hamiltonian function can be viewed as being composed of two operations. First, a Lagrangian submanifold of the cotangent bundle of the phase space is generated and then it is transferred to the tangent bundle with the help of the natural isomorphism of these bundles. We describe a generalization of the first phase of the generation mechanism. The symplectic structure of the phase space is not used.

Let P be the phase space of a mechanical system. Let C be a submanifold of P and let K be a vector subbundle of the bundle $\mathsf{T}_C P$. The *polar*

$$K^\circ = \{f \in \mathsf{T}^*P;\, p = \pi_P(f) \in C, \langle u, f\rangle = 0 \text{ for each } u \in K_p\} \tag{66}$$

is a vector subbundle of the bundle $\mathsf{T}^*_C P$. Let $\varphi: C \to K^*$ be a differentiable section of the bundle K^*.

Proposition 6. The set

$$N = \{f \in \mathbf{T}^*P;\ p = \pi_P(f) \in C,\ \langle u, f\rangle = \langle u, \varphi\rangle \text{ for each } u \in K_p\} \tag{67}$$

is an affine subbundle of the vector bundle $\mathbf{T}^*_C P$, modelled on the vector subbundle K°.

Proof. For each $p_0 \in C$, there is a neighbourhood U of p_0 in C and a local linear trivialization $\eta: \mathbf{T}^*_U P \to \mathbb{R}^m$ of the bundle $\mathbf{T}^*_C P$, such that for each $p \in U$

$$K^\circ_p = \{f \in \mathbf{T}^*_p P;\ y_\mu(f) = 0 \text{ for } \mu = k+1, \dots, k+m\} \tag{68}$$

where the coordinates $y_\mu: \mathbf{T}^*_U P \to \mathbb{R}$ are the compositions $pr_\mu \circ \eta$ of η with the canonical projections $pr_\mu: \mathbb{R}^m \to \mathbb{R}$. For each $p_0 \in C$, there is a neighbourhood U of p_0 in C and a differentiable section $\tilde{\varphi}: U \to \mathbf{T}^*_U P$ of the bundle $\mathbf{T}^*_U P$, such that $\langle u, \tilde{\varphi}\rangle = \langle u, \varphi\rangle$ for each $u \in K_U$. Let U be a neighbourhood of p_0 such that both the trivialization η and the section $\tilde{\varphi}$ exist. The mapping

$$\begin{aligned} \eta' &: \mathbf{T}^*_U P \to \mathbb{R}^m \\ &: f \mapsto \eta(f - \tilde{\varphi}(\pi_P(f))) \end{aligned} \tag{69}$$

is a local affine trivialization of the bundle $\mathbf{T}^*_C P$, such that for each $p \in U$

$$N_p = \{f \in \mathbf{T}^*_p P;\ y'_\mu(f) = y_\mu(f - \tilde{\varphi}(\pi_P(f))) = 0 \text{ for } \mu = k+1, \dots, k+m\}. \tag{70}$$

It follows that N is an affine subbundle of $\mathbf{T}^*_C P$. If f and f' are elements of N_p, then $f' - f \in K^\circ_p$. Hence, K° is the model bundle for N. □

The set N in the above proposition is said to be *generated* by the triple (C, K, φ).

8. Integrability of implicit differential equations in a symplectic phase space

Let (P, ω) be the symplectic phase space of a mechanical system and let $\beta: \mathbf{T}P \to \mathbf{T}^*P$ be the natural isomorphism provided by the symplectic structure. Let $V \subset \mathbf{T}_p P$ be a vector subspace. We denote by $V^¶$ the symplectic polar

$$\beta^{-1}(V^\circ) = \{w \in \mathbf{T}_p P;\ \langle u \wedge w, \omega\rangle = 0 \text{ for each } u \in V\}. \tag{71}$$

Let C be a submanifold of P, let K be a vector subbundle of the bundle $\mathbf{T}_C P$ and let $\varphi: C \to K^*$ be a differentiable section of the fibration $\lambda: K^* \to C$. The inverse image $\beta^{-1}(N)$ of the set N, defined by (67), is the implicit differential equation

$$D = \{w \in \mathbf{T}P;\ p = \tau_P(w) \in C,\ \langle u \wedge w, \omega\rangle = \langle u, \varphi\rangle \text{ for each } u \in K_p\}. \tag{72}$$

The set D is an affine subbundle of the vector bundle $\mathbf{T}_C P$, modelled on the subbundle $K^¶ = \beta^{-1}(K^\circ)$.

A Dirac system

$$D = \{w \in \mathbf{T}P;\ p = \tau_P(w) \in C,\ \langle u \wedge w, \omega\rangle = \langle u, \mathrm{d}H\rangle \text{ for each } u \in \mathbf{T}_p C\} \tag{73}$$

is a special case of (72), obtained by setting $K = \mathsf{T}C$ and $\varphi = \mathrm{d}H$, where $H: C \to \mathbb{R}$ is the Hamiltonian function. The pair (C, H) represents the Dirac system (73).

The integrability condition $D \subset \mathsf{T}C$ is satisfied if, and only if, the condition $D_p \subset \mathsf{T}_pC$ is satisfied for each $p \in C$. The set

$$D_p = \{w \in \mathsf{T}_pP;\ \langle u \wedge w, \omega\rangle = \langle u, \varphi\rangle \text{ for each } u \in K_p\} \tag{74}$$

is an affine subspace of T_pP modelled on the vector subspace

$$\{w \in \mathsf{T}_pP;\ \langle u \wedge w, \omega\rangle = 0 \text{ for each } u \in K_p\} = K_p^{\P}. \tag{75}$$

For the space T_pC, we use the representation

$$\begin{aligned}\mathsf{T}_pC &= (\mathsf{T}_p^{\P}C)^{\P}\\ &= \{w \in \mathsf{T}_pP;\ \langle u \wedge w, \omega\rangle = 0 \text{ for each } u \in \mathsf{T}_p^{\P}C\}.\end{aligned} \tag{76}$$

We denote by $\ker\varphi(p)$ the space

$$\{u \in \mathsf{T}_pP;\ u \in K_p,\ \langle u, \varphi\rangle = 0\}. \tag{77}$$

Proposition 7. If $\mathsf{T}_p^{\P}C \subset \ker\varphi(p)$, then $D_p \subset \mathsf{T}_pC$.

Proof. Let $w \in D_p$. From (74), we have

$$\langle u \wedge w, \omega\rangle = \langle u, \varphi\rangle \tag{78}$$

for each $u \in K_p$. If $\mathsf{T}_p^{\P}C \subset \ker\varphi(p)$, then it follows that

$$\langle u \wedge w, \omega\rangle = 0 \tag{79}$$

for each $u \in \mathsf{T}_p^{\P}C$. Hence, $w \in (\mathsf{T}_p^{\P}C)^{\P} = \mathsf{T}_pC$. □

Lemma 1. If the intersection $D_p \cap \mathsf{T}_pC$ is not empty, then $K_p \cap \mathsf{T}_p^{\P}C \subset \ker\varphi(p)$.

Proof. Let $D_p \cap \mathsf{T}_pC$ be not empty and let $w \in D_p \cap \mathsf{T}_pC$. If $u \in K_p \cap \mathsf{T}_p^{\P}C$, then

$$\langle u \wedge w, \omega\rangle = \langle u, \varphi\rangle \tag{80}$$

from (74), and

$$\langle u \wedge w, \omega\rangle = 0 \tag{81}$$

from (76). Hence, $\langle u, \varphi\rangle = 0$. □

Proposition 8. If $D_p \subset \mathsf{T}_pC$, then $\mathsf{T}_p^{\P}C \subset \ker\varphi(p)$.

Proof. Let $v \in K_p^{\P}$. If $w \in D_p$, then $w + v \in D_p$. If $D_p \subset \mathsf{T}_pC$, then it follows that $w \in \mathsf{T}_pC$ and $w + v \in \mathsf{T}_pC$. Hence, $v \in \mathsf{T}_pC$. We have established the inclusion $K_p^{\P} \subset \mathsf{T}_pC$ equivalent to $K_p \subset \mathsf{T}_p^{\P}C$. If $D_p \subset \mathsf{T}_pC$, then the intersection $D_p \cap \mathsf{T}_pC$ is not empty. If $u \in \mathsf{T}_p^{\P}C$, then $u \in K_p$. From lemma 1, it follows that $u \in \ker\varphi(p)$. Hence, $\mathsf{T}_p^{\P}C \subset \ker\varphi(p)$. □

The following corollary summarizes the results obtained so far.

Corollary 1. The implicit differential equation

$$D = \{w \in \mathbf{T}P;\, p = \tau_P(w) \in C, \langle u \wedge w, \omega\rangle = \langle u, \varphi\rangle \text{ for each } u \in K_p\} \quad (82)$$

is integrable if, and only if, $\mathbf{T}_p^\P C \subset \ker\varphi(p)$ for each $p \in C$.

For the special case of a Dirac system, we obtain the following known theorem.

Theorem 1. A Dirac system

$$D = \{w \in \mathbf{T}P;\;\, p = \tau_P(w) \in C, \langle u \wedge w, \omega\rangle = \langle u, \mathrm{d}H\rangle \text{ for each } u \in \mathbf{T}_pC\} \quad (83)$$

is integrable if, and only if, the submanifold $C \subset P$ is co-isotropic and the Hamiltonian function $H: C \to \mathbb{R}$ is constant on leaves of the characteristic foliation of C.

We recall that a submanifold $C \subset P$ is said to be *co-isotropic* if $\mathbf{T}^\P C \subset \mathbf{T}C$. The set $\mathbf{T}^\P C$ is called the *characteristic distribution* of a co-isotropic submanifold $C \subset P$. The characteristic distribution is Frobenius integrable. Its integral foliation is called the *characteristic foliation* of C. In Dirac's terminology, a co-isotropic submanifold of the phase space is a *first-class constraint*. A submanifold $C \subset P$ is called a *second-class constraint* if the symplectic form ω, restricted to C, is non-degenerate. For an intrinsic definition of the class of a submanifold of a symplectic manifold, see [6].

9. Extracting the integrable part of a Dirac system

Proposition 9. For the set D of formula (72), we have

$$\tau_P(D \cap \mathbf{T}C) \subset \{p \in P;\;\, p \in C, K_p \cap \mathbf{T}_p^\P C \subset \ker\varphi(p)\}. \quad (84)$$

Proof. If $p \in \tau_P(D \cap \mathbf{T}C)$, then $p \in C$ and the intersection $D_p \cap \mathbf{T}_pC$ is not empty. It follows from lemma 1 that $K_p \cap \mathbf{T}_p^\P C \subset \ker\varphi(p)$. □

We denote the set $\{p \in P; p \in C, K_p \cap \mathbf{T}_p^\P C \subset \ker\varphi(p)\}$ by C'. We define the set $K' = \cup_{p\in C'} K'_p$, where $K'_p = K_p + \mathbf{T}_p^\P C$, and the mapping $\varphi': C' \to K'^*$, characterized by

$$\langle u, \varphi'(p)\rangle = \begin{cases} \langle u, \varphi(p)\rangle & \text{if } u \in K_p \\ 0 & \text{if } u \in \mathbf{T}_p^\P C. \end{cases} \quad (85)$$

Proposition 10. If $p \in C'$, then

$$D_p \cap \mathbf{T}_pC = \{w \in \mathbf{T}_pP; \langle u \wedge w, \omega\rangle = \langle u, \varphi'\rangle \text{ for each } u \in K'_p\}. \quad (86)$$

Proof. Let $p \in C'$. It follows from (74) and (76) that $w \in D_p \cap \mathbf{T}_pC$ if, and only if,

$$\langle u \wedge w, \omega\rangle = \begin{cases} \langle u, \varphi\rangle & \text{if } u \in K_p \\ 0 & \text{if } u \in \mathbf{T}_p^\P C. \end{cases} \quad (87)$$

From the definition of φ', it follows that (87) is equivalent to

$$\langle u \wedge w, \omega\rangle = \langle u, \varphi\rangle \quad (88)$$

for each $u \in K_p + \mathbf{T}_p^\P C = K'_p$. □

Proposition 11. For the set D of formula (72), we have

$$\tau_P(D \cap \mathbf{T}C) \supset C' = \{p \in P;\; p \in C, K_p \cap \mathbf{T}^{\P}C \subset \ker \varphi(p)\}. \tag{89}$$

Proof. If $p \in C'$, then it follows from proposition 10 that $D_p \cap \mathbf{T}_pC$ is not empty. Hence, $p \in \tau_P(D \cap \mathbf{T}C)$. □

We have obtained the representation

$$D' = \{w \in \mathbf{T}P;\; p = \tau_P(w) \in C', \langle u \wedge w, \omega\rangle = \langle u, \varphi'\rangle \text{ for each } u \in K'_p\} \tag{90}$$

for the set $D' = D \cap \mathbf{T}C$. This representation is the basis of the extraction algorithm for a Dirac system.

We start with a Dirac system $D \subset \mathbf{T}P$ generated by a Hamiltonian function $H: C \to \mathbb{R}$ defined on a submanifold $C \subset P$. We introduce a sequence of objects

$$(C^0, K^0, \varphi^0), (C^1, K^1, \varphi^1), \ldots, (C^k, K^k, \varphi^k), \ldots \tag{91}$$

where

$$C^0 = C \qquad K^0 = K \qquad \varphi^0 = \varphi \tag{92}$$

$$C^k = \{p \in P; p \in C^{k-1}, K_p^{k-1} \cap \mathbf{T}_p^{\P}C^{k-1} \subset \ker \varphi^{k-1}(p)\} \tag{93}$$

$$K^k = \cup_{p \in C^k}(K_p^{k-1} + \mathbf{T}_p^{\P}C^{k-1}) \tag{94}$$

and φ^k is characterized by

$$\langle u, \varphi^k(p)\rangle = \begin{cases} \langle u, \varphi^{k-1}(p)\rangle & \text{if } u \in K_p^{k-1} \\ 0 & \text{if } u \in \mathbf{T}_p^{\P}C^{k-1}. \end{cases} \tag{95}$$

It is assumed that sets C^k are submanifolds of P and K^k are subbundles of $\mathbf{T}_{C^k}P$. Since the dimension of P is finite, sequence (91) is constant, starting with some index $k = m$. The integrable implicit equation D^m, generated by (C^m, K^m, φ^m), is the integrable part of the original Dirac system.

Example 10. Let the space $\bar{P} = \mathbb{R}^4$ of example 8 be given a symplectic structure with the symplectic form ω defined by

$$\langle (x, p, r, s, \dot{x}, \dot{p}, \dot{r}, \dot{s}) \wedge (x, p, r, s, \delta x, \delta p, \delta r, \delta s), \omega\rangle = \dot{r}\delta x + \dot{s}\delta p - \dot{x}\delta r - \dot{p}\delta s. \tag{96}$$

The implicit differential equation (47) is a Dirac system $(\bar{C}, \bar{H})$ with

$$\bar{C} = \{(x, p, r, s) \in \bar{P}; r = p, s = 0\} \tag{97}$$

and

$$\begin{aligned} \bar{H}&: \bar{C} \to \mathbb{R} \\ &: (x, p, r, s) \mapsto H(x, p). \end{aligned} \tag{98}$$

The set $\bar{C}$ is a second-class constraint.

The extraction algorithm starts with

$$C^0 = \bar{C} \tag{99}$$

$$\begin{aligned} K^0 &= \mathbf{T}C^0 \\ &= \{(x, p, r, s, \delta x, \delta p, \delta r, \delta s) \in \mathbf{T}\bar{P}; r = p, s = 0, \delta r = \delta p, \delta s = 0\} \end{aligned} \tag{100}$$

and

$$\varphi^0 : C^0 \to K^{0*} \tag{101}$$

characterized by

$$\langle (x, p, r, s, \delta x, \delta p, \delta r, \delta s), \varphi^0 \rangle = \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial p}\delta p. \tag{102}$$

The set $\mathbf{T}^{\P}C^0$ is the set of all $(x, p, r, s, \delta x, \delta p, \delta r, \delta s) \in \mathbf{T}\bar{P}$, such that $r = p, s = 0$ and

$$\dot{r}\delta x + \dot{s}\delta p - \dot{x}\delta r - \dot{p}\delta s = 0 \tag{103}$$

for all $(x, p, r, s, \dot{x}, \dot{p}, \dot{r}, \dot{s}) \in \mathbf{T}C^0$. It follows that

$$\mathbf{T}^{\P}C^0 = \{(x, p, r, s, \delta x, \delta p, \delta r, \delta s) \in \mathbf{T}\bar{P}; r = p, s = 0, \delta s - \delta x = 0, \delta r = 0\}. \tag{104}$$

The set $K^0 \cap \mathbf{T}^{\P}C^0$ contains only zero vectors. This confirms the observation that $\bar{C}$ is a second-class constraint. The condition

$$K^0_{(x,p,r,s)} \cap \mathbf{T}^{\P}_{(x,p,r,s)}C^0 \subset \ker \varphi^0(x, p, r, s) \tag{105}$$

is trivially satisfied for all points $(x, p, r, s) \in C^0$. Hence, $C^1 = C^0$. The set

$$K^1 = \cup_{(x,p,r,s)\in C^0}(K^0_{(x,p,r,s)} + \mathbf{T}^{\P}_{(x,p,r,s)}C^0) \tag{106}$$

is easily seen to coincide with $\mathbf{T}_{C^0}\bar{P}$ and

$$\varphi^1 : C^1 \to K^{1*} \tag{107}$$

is characterized by

$$\langle (x, p, r, s, \delta x, \delta p, \delta r, \delta s), \varphi^1 \rangle = \frac{\partial H}{\partial x}(\delta x - \delta s) + \frac{\partial H}{\partial p}\delta r. \tag{108}$$

The triple (C^1, K^1, φ^1) generates the implicit differential equation (51) of example 8.

Example 11. Let the space $P = \mathbb{R}^4$ of example 9 be given a symplectic structure with the symplectic form ω, defined by

$$\langle (x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \wedge (x, y, p, q, \delta x, \delta y, \delta p, \delta q), \omega \rangle = \dot{p}\delta x + \dot{q}\delta y - \dot{x}\delta p - \dot{y}\delta q. \quad (109)$$

Implicit differential equation (57) is a Dirac system (C, H) with

$$C = \{(x, y, p, q) \in P; x^2 + y^2 \neq 0, xp + yq = 0\} \quad (110)$$

and

$$\begin{aligned} H&: C \to \mathbb{R} \\ &: (x, y, p, q) \mapsto \frac{p^2 + q^2}{2k(x^2 + y^2)}. \end{aligned} \quad (111)$$

The extraction algorithm starts with

$$C^0 = C \quad (112)$$

$$\begin{aligned} K^0 = \mathbf{T}C^0 = \{&(x, y, p, q, \delta x, \delta y, \delta p, \delta q) \in \mathbf{T}P; x^2 + y^2 \neq 0, xp + yq = 0, \\ &\delta xp + x\delta p + \delta yq + y\delta q = 0\} \end{aligned} \quad (113)$$

and

$$\varphi^0 : C^0 \to K^{0*} \quad (114)$$

characterized by

$$\langle (x, y, p, q, \delta x, \delta y, \delta p, \delta q), \varphi^0 \rangle = \frac{p\delta p + q\delta q}{k(x^2 + y^2)} - \frac{(p^2 + q^2)(x\delta x + y\delta y)}{k(x^2 + y^2)^2}. \quad (115)$$

We have

$$\begin{aligned} \mathbf{T}^{\P}C^0 = \Big\{&(x, y, p, q, \delta x, \delta y, \delta p, \delta q) \in \mathbf{T}P; x^2 + y^2 \neq 0, xp + yq = 0, \\ &\delta x = \frac{x\delta x + y\delta y}{x^2 + y^2}x, \delta y = \frac{x\delta x + y\delta y}{x^2 + y^2}y, \\ &\delta p = -\frac{x\delta x + y\delta y}{x^2 + y^2}p, \delta q = -\frac{x\delta x + y\delta y}{x^2 + y^2}q \Big\}. \end{aligned} \quad (116)$$

It is easily seen that $\mathbf{T}^{\P}C^0 \subset \mathbf{T}C^0$. It follows that C is a first-class constraint. We have $K^0 \cap \mathbf{T}^{\P}C^0 = \mathbf{T}^{\P}C^0$ and the equation

$$\frac{p\delta p + q\delta q}{k(x^2 + y^2)} - \frac{(p^2 + q^2)(x\delta x + y\delta y)}{k(x^2 + y^2)^2} = 0 \quad (117)$$

with

$$(x, y, p, q, \delta x, \delta y, \delta p, \delta q) \in \mathbf{T}^{\P}C^0 \quad (118)$$

yields

$$-2\frac{x\delta x + y\delta y}{k(x^2 + y^2)^2}(p^2 + q^2) = 0. \quad (119)$$

Hence,

$$C^1 = \{(x, y, p, q) \in P; x^2 + y^2 \neq 0, p = 0, q = 0\}. \quad (120)$$

The algorithm terminates with

$$K^1 = \{(x, y, p, q, \delta x, \delta y, \delta p, \delta q) \in \mathbf{T}P; x^2 + y^2 \neq 0, p = 0, q = 0, x\delta p + y\delta q = 0\} \quad (121)$$

and $\varphi^1 = 0$. The triple (C^1, K^1, φ^1) generates the integrable differential equation (61) of example 9.

References

[1] Dirac P A M 1950 Generalized Hamiltonian dynamics *Canad. J. Math.* 2 129–48; 1967 *Lectures in Quantum Mechanics* Belfer Graduate School of Science, Yeshiva University, New York
Anderson J L and Bergmann P G 1951 Constraints in covariant field theories *Phys. Rev.* 83 1018–26

[2] Tulczyjew W M 1989 *Geometrical Formulation of Physical Theories* (Naples: Bibliopolis)

[3] Arnold V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Berlin: Springer)

[4] Marmo G, Mendella G and W M Tulczyjew 1992 Symmetries and constants of the motion for dynamics in implicit form *Ann. Inst. H. Poincaré* 57 147–66

[5] Haag R 1952 Der kanonische formalismus in entarteten fällen *Z. Angew. Math. Mech.* 32 197–202

[6] Menzio M R and Tulczyjew W M 1978 Infinitesimal symplectic relations and generalized Hamiltonian dynamics *Ann. Inst. H. Poincaré* 28 349–67